Problem Solving Marathon (11/3/08)

Semicircle in a square (153)

Find the area of the largest semicircle that can be inscribed in the unit square.

Folded sheet of paper (1)

A rectangular sheet of paper is folded so that two diagonally opposite corners come together. If the crease formed is the same length as the longer side of the sheet, what is the ratio of the longer side of the sheet to the shorter side?

Ant on a box (16)

A 12 by 25 by 36 cm cereal box is lying on the floor on one of its 25 by 36 cm faces. An ant, located at one of the bottom corners of the box, must crawl along the outside of the box to reach the opposite bottom corner. What is the length of the shortest such path?

Note: The ant can walk on any of the five faces of the box, except for the bottom face, which is flush in contact with the floor. It can crawl along any of the edges. It cannot crawl under the box.

Zero-sum game (16)

Two players take turns choosing one number at a time (without replacement) from the set $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. The first player to obtain three numbers (out of three, four, or five) which sum to 0 wins.

Does either player have a forced win?

Three children

On the first day of a new job, a colleague invites you around for a barbecue. As the two of you arrive at his home, a young boy throws open the door to welcome his father. “My other two kids will be home soon!” remarks your colleague.

Waiting in the kitchen while your colleague gets some drinks from the basement, you notice a letter from the principal of the local school tacked to the noticeboard. “Dear Parents,” it begins, “This is the time of year when I write to all parents, such as yourselves, who have a girl or girls in the school, asking you to volunteer your time to help the girls' soccer team.” “Hmmm,” you think to yourself, “clearly they have at least one of each!”

This, of course, leaves two possibilities: two boys and a girl, or two girls and a boy. Are these two possibilities equally likely, or is one more likely than the other?

Note: This is not a trick puzzle. You should assume all things that it seems you're meant to assume, and not assume things that you aren't told to assume.

Solution to puzzle 17: Three children

We assume that each birth is an independent event, for which the probability of a boy is the same as the probability of a girl. There are, then, three possibilities for your colleague's family, all equally likely:

- Boy, Boy, Girl
- Boy, Girl, Boy
- Boy, Girl, Girl

Therefore there is a 2/3 chance that the colleague has two boys and a girl, and a 1/3 chance he has two girls and a boy.

Remarks

Note that in each of the *ordered triples* above, (BBG, BGB, BGG), the first letter represents the gender of the *first* child. The context of the word *first* may be chosen at our convenience. For example, it may be the eldest child (if we sort by descending age), the shortest (sort by ascending height), or perhaps the child whose forename is first alphabetically. In this case, based upon the information we are given, the context we choose is “the first child we meet.” The second and third letters in each ordered triple represent the genders of the other two children, neither of whom we have met.

Another way to arrive at the answer is to note that the boy who opens the door is essentially a red herring. Leaving him aside, the puzzle asks us to compare the probabilities that the other two children are (a) both girls, or (b) one girl and one boy. (The letter from the principal is carefully worded to leave both options open.) The second option (older sister, younger brother, or older brother, younger sister) is twice as likely as the first (elder sister, younger sister.)

As with many conditional probability questions, this result may seem surprising at first. If you are skeptical, I urge you to carry out a simulation, either manually or programatically. Setting up such a simulation forces you to analyze exactly what is happening.

The situation can be modelled by throwing three coins. Let heads represent boys, and tails girls. One coin in each throw should be set aside to be checked whether it is a boy or a girl. (This corresponds to the boy who opens the door.) It needn't be the same coin each time (though it could be), and it needn't be thrown first (though it might be), but it must be chosen independently of whether it shows heads or tails.

One convenient method would be to throw one dime and two nickels. If the dime (first child) shows heads (a boy) and one or both of the nickels shows tails (at least one girl), then we have a faithful representation of the puzzle situation. Other scenarios are discarded. Of the scenarios retained, in roughly two out of three cases the three coins will show two heads.

Solution to puzzle 16: Zero-sum game

Two players take turns choosing one number at a time (without replacement) from the set $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. The first player to obtain three numbers (out of three, four, or five) which sum to 0 wins.

Does either player have a forced win?

Consider a 3×3 magic square, wherein all of the rows, columns, and diagonals sum to 0; example below. It's not difficult to see that the aim of the game, as stated, can be satisfied if, and only if, the three integers fall in the same row, column, or diagonal.

$$\begin{array}{ccc} 1 & 2 & -3 \\ -4 & 0 & 4 \\ 3 & -2 & -1 \end{array}$$

Hence the game is equivalent to tic-tac-toe, or noughts and crosses, a game which, with best play, is well known to be a draw.

Therefore neither player has a forced win.

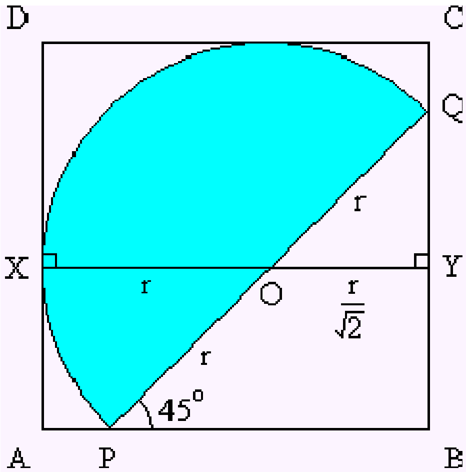
Solution to puzzle 153: Semicircle in a square

Find the area of the largest semicircle that can be inscribed in the unit square.

It is clear that the largest semicircle will touch the sides of the square at both ends of its diameter, and will also be tangent to the perimeter.

One obvious solution is a semicircle whose diameter coincides with one side of the square.
Such a semicircle will have radius = 1/2 and area = $\pi(1/2)^2/2 = \pi/8 \approx 0.3927$.
Can we do better?

Consider a semicircle whose diameter endpoints touch two adjacent sides of the square. It is intuitively obvious that such a semicircle of maximal area will be tangent to both of the other sides of the square, but see the remarks below for a more rigorous justification.



Since the figure is symmetrical in the diagonal BD, $\angle QPB = 45°$.

Consider the point X on AD at which the semicircle is tangent to AD. A line extended from X that is perpendicular to the tangent will be parallel to AB, and will also pass through the middle of the semicircle diameter. Let the line meet BC at Y.

$OY = r \cos 45° = r/\sqrt{2}$.
Hence $1 = AB = r + r/\sqrt{2} = r(1 + 1/\sqrt{2})$.
Thus $r = 1/(1 + 1/\sqrt{2})$.
Rationalizing the denominator, we obtain $r = 2 - \sqrt{2}$and area $= \pi r^2/2 = \pi(3 - 2\sqrt{2})$.

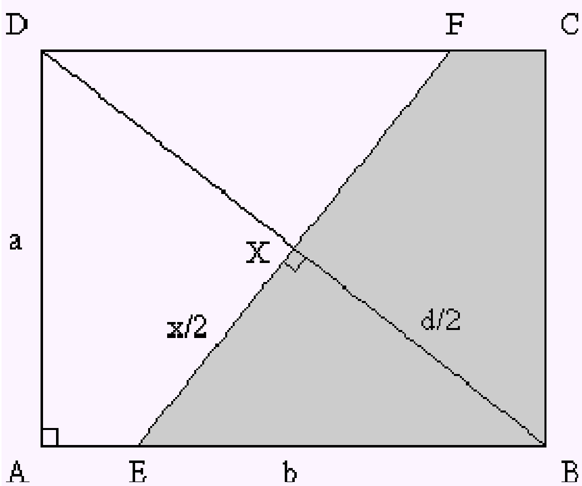
Thus, the area of the largest semicircle that can be inscribed in the unit square is $\pi(3 - 2\sqrt{2}) \approx 0.539$

Solution to puzzle 1: Folded sheet of paper

We will find the length of the fold in terms of the dimensions of the sheet of paper, and set this equal to the length of the longer side.

Let the sheet of paper be ABCD, and have sides AD = a, AB = b, where $a \leq b$. Let the fold line be EF, of length x. Let d the length of the diagonal.
By Pythagoras' Theorem, $d^2 = a^2 + b^2$.

Draw straight line BD between the two corners used to make the fold. It's clear by symmetry that this diagonal intersects the fold at right angles. Further, also by symmetry, both lines meet at the center of the rectangle, X, and bisect each other.



Triangles DAB and XEB contain two common angles, and therefore are similar.
Hence $a/b = (x/2)/(d/2) = x/d$.
Therefore $x = (a/b)\sqrt{(a^2 + b^2)}$.

If x = b, as we require, then $a^2(a^2 + b^2) = b^4$, and so $b^4 - a^2b^2 - a^4 = 0$.

Solving as a quadratic equation in b^2, we have $b^2 = [a^2 \pm \sqrt{(a^4 + 4a^4)}]/2$.

$= a^2 (1 \pm \sqrt{5})/2$.

Rejecting the negative roots, b/a = ratio of longer side to shorter side = $\sqrt{\frac{1+\sqrt{5}}{2}}$

